Instrumented techniques in tool – and object perspectives

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The aim of this paper is to report from a study of the role of instrumented techniques in the students' learning process. The paper analyses an episode from a case study of students solving differential equations in a CAS environment. The analysis demonstrates how tasks can be designed with the aim to encourage the students to change between the perspective of tool on a mathematical conception and the perspective of object on the conception. Reasons are given in the paper for the assertion, that changing between these two perspectives supports the instrumental genesis as well as the conceptual development.

Instrumental genesis and instrumented techniques

The French theory of instrumental genesis is based on the idea that an artefact, for example a CAS calculator, does not in itself serve as a tool for the student. It becomes a tool, referred to as an *instrument* in this notion, only by the student's formation of (one or more) mental utilisation scheme(s). The term *instrumental genesis* denotes the process in which the artefact becomes an instrument. (Drijvers and Gravemeijer 2005 pp 165-169). The formation of utilisation schemes and the building up of instrumented action schemes proceed through activities in 'The two-sided relationship between tool and learner as a process in which the tool in a manner of speaking shapes the thinking of the learner, but also is shaped by his thinking'. (ibid. p 190). The French framework is underlying the theory of instrumental genesis: according to Luc Trouche the scheme concept, encompassing utilisation schemes and instrumented action schemes, was introduced by G. Vergnaud as 'an invariant organization of activity for a given class of situations. It has an intention and a goal and constitutes a functional dynamic entity. In order to understand its function and dynamic, one has to take into account its components as a whole: goal and subgoals, anticipations, rules of action, of gathering information and exercising control, operational invariants and possibilities of inference within the situation. (Trouche 2005 p 149)

The formation of utilisation schemes and instrumental action schemes, thereby, is pivotal for the instrumental genesis. Since the utilisation schemes are mental, they are not directly accessible for study and analysis. The concept of *instrumented techniques*, taken as the external, visible and manifest part of the instrumented action scheme, therefore, is of special interest. An instrumented technique is 'a set of rules and methods in a technological environment that is used for solving a specific type of problem.' (Drijvers & Gravemeijer 2005 p 169). An instrumented technique includes conceptual elements as far as the technique reflects the schemes. This leads me to two crucial points:

- A student's development of an instrumented action scheme can be studied by inquiry of the student's development and use of instrumented techniques related to the scheme.
- Development of mathematical conceptions cannot be studied if use of technology is considered separate from the student's other activities.

The first point stresses the importance of empirical studies of students' work. The second point opposes my research to the standpoint, that teaching may be performed independently of what tools the students have at their disposal. This is in line with Jean-Baptiste Lagrange who stressed, that 'the traditional opposition of concepts and skills should be tempered by recognising a technical dimension in mathematical activity, which is not reducible to skills. A cause of misunderstanding is that, at certain moments, a technique can take the form of a skill.' (Lagrange 2005 pp 131-132).

Tool – and object perspectives.

During a recently concluded research project (Andresen 2005) on the teaching of differential equations in upper secondary school in laptop-classes, I have constructed and tested a conceptual tool, flexibility. This notion of flexibility encompasses the tool – and object perspectives subject to this paper. In the following, the definition of flexibility is reproduced without further explanations. For a discussion of details and examples, see (Andresen 2005). Definition: *The flexibility of a mathematical conception constructed by a person is the designation of all the changes of perspective and all the changes between different representations the person can manage within this conception.* The changes of perspective considered are divided in three groups:

a) Dualities of perspectives intrinsic to mathematics: 1. Local – global, 2. General – specific 3. Analytic- constructive

b) Dualities of perspectives linked to the construction of epistemic knowledge: 4. Process - object, 5. Situated – decontextualised

c) Dualities of perspectives linked to the construction of pragmatic knowledge: 6. Tool – object, 7. Model – reality, 8. Model of - model for

Three main representations are considered: graphic representation, analytic representation (or formal language), and natural language. A fourth, called technical representation (or computer language) is included as well, caused by the use of laptops. There is no symmetry between the four representations.

The conceptual tool flexibility serves to capture and conceptualise certain learning potentials experienced by teachers and students, for instance, when using the laptops in a modelling-context. One element of flexibility with special relevance for the theme of this paper is changes between a *tool perspective* on mathematical conceptions and an *object perspective* on the same conception. The notion of a tool perspective on a mathematical conception is opposed to a pure skill understanding of mathematical activity and the notion includes the technical dimension mentioned by Lagrange. The duality composed by a tool perspective on a mathematical conception and an object perspective on the same conception appears to be appropriate in problem-solving settings, in the same way as Anna Sfard's process – object duality (Sfard 1991) is useful to frame aspects of learning mathematics. The term *tool perspective* here refers to the mathematical processes, carried out to serve a concrete purpose. This resembles the use of the term *tool* synonymously with *instrument* in contrast to *artefact*. This notion of tool perspective on a given mathematical conception is in accordance with Régine Douady's definition: 'We say that a concept is a tool when the interest is focused on its use for solving a problem. A tool is involved in a specific context, by somebody, at a given time. A given tool may be adapted to several problems; several tools may be adapted to a given problem.' (Douady 1991 p 115)

The distinction between the pair of *process – and object perspective* and the pair *of tool – and object perspective* can be illustrated by the following example: a tool perspective on the conception of derivative of a function could be the derivative seen and used as a means for finding out how the function changes over time. The corresponding object perspective could be the derivative, characterised or categorised by its merits and demerits when it was assessed in the context of solving a specific problem. In contrast, a process perspective of derivative could be focusing on the actual determination or calculation of the derivative in question. The corresponding object perspective could be the derivative, generally characterised or categorised by its qualities within in a structure of functions. Mathematical activities, then, are considered from a tool perspective when they are part of a (problem solving) technique, regardless of its being instrumented or not. The generation of the instrument, then, is in a crucial way linked to the change to object perspective: From the object perspective corresponding to a tool perspective, a unit is considered which may encompass intension, goal, conditions and prerequisites, restrictions, function and dynamics. Like in the case of process – object, the object perspective implies an encapsulation of the conception as a tool. So for the student, the development of an object perspective gradually leads to master the techniques in which the conception is embedded and to complete the formation of the connected instrumented schemes.

Change of perspective to support learning

Basic to the research, which lead to the construction of the conceptual tool *flexibility* was the idea that learning is supported by alternating diving into the process of solving a problem and taking a distant look upon the activities and efforts (Andresen 2004). Edith Ackermann presents this idea in (Ackermann 1990) as a mean to integrate, roughly speaking, Jean Piaget's and Seymour Papert's views on children's cognitive development. In her paper, Ackerman combines the Piaget'ian construction of invariables with Papert's situated learning in her dynamical approach to cognitive growth. *Flexibility* incorporates this basic idea in the form of the aforementioned changes within dualities of perspectives on given mathematical conceptions. One aim of the research project was to inquire how the teacher can provoke and support the students' change of perspective in both directions within these dualities, and to interpret the role of such changes for the students' ongoing mathematical activities.

In the actual case, a group of three students used several instrumented techniques during three episodes. The episodes were analysed and interpretations of the techniques' role in the students' learning process are presented.

Case

The case presented in this paper is part of the data from my Ph.D. project. These part of the project's data were produced from a small scale, qualitative inquiry which encompassed classroom observations in four classes, 50 lessons in all, field notes, students' written reports and teaching materials prepared for a sequence of teaching differential equations from a dynamical point of view using the software Derive.

A group of three students were working with a differential equation model of the transformation of cholesterol in the human body. The students were in third year of an experimental class in upper secondary school, where all the students had their own laptops at their disposal from first year on. The CAS software Derive was installed on the laptops. This case is based on group's work during one lesson which was video recorded. The students' written report and the teaching materials were examined in relation to the analysis of the case.

The students were preparing a written report on a series of tasks, which concerned exploring a model for transformation of cholesterol, presented in the textbook. The tasks aimed to stimulate the students' learning about *equilibrium point* and *general* as well as *specific solutions to differential equations*. Further, the tasks concerned *relations between general and specific solution* and *connections between analytic and graphic representation*, both mediated by computer language. In the case, the group was in an early phase of their work, concentrating on this text from the teaching materials (Hjersing et.al. 2004):

> ... another handy form is:
>
> $$\frac{dC}{dt} = 0.1(265 - C) \qquad (8.2)$$
>
> (Bubba changes his diet at $t_0 = 0$, with $C_0 = 180$ mg/dl, the new daily cholesterol intake is E = 250 mg/day.)
>
> If we let t0 = 0 be the time where Bubba starts eating at the grill and if Bubba's level of cholesterol at that time is supposed to be C0 = 180 mg/dl, then Bubba's cholesterol level is expressed:

$$\frac{dC}{dt} = 0.1(265 - C)$$
$$C(0) = 180 \qquad (8.3)$$

Tasks

1. Find the equilibrium point for (8.2) and analyse the variation of the sign of the right side.

Is the equilibrium point a sink or a source? Use the answers to sketch (in hand) more solutions to this differential equation.

2. Find the general solution to the differential equation (8.2) (Show calculations)

3. Find the specific solution to the initial value problem (8.3)

4. If Bubba keeps this high cholesterol level diet for a very long time (one year or more), at what level will he end? Explain how you reach the conclusion?

During the case, the students used several instrumented techniques: First, in episode 1, they used the Derive command RK[1] to obtain a graph of the solution to the differential equation (8.2) with the initial conditions t0=0, $C(t_0)=180$. In episode 2, they used their compendium of formulas supplied by paper and pencil techniques to find the general solution to the equation. The solution was typed into the computer and the students used the Derive command VECTOR to get a family of graphs of solution curves, as kind of an intermediate between general and specific solution. To answer the next question, they substituted the initial values in the formula for the general solution, calculated the constant d (determined by the initial values) and substituted it into the expression. To answer question 4. in episode 3, the students repeated graphing the same solution curve as they graphed in the first episode, but this time based on the expression obtained from the preceding answer. Their answer to question 4., then, was based on visual inspection of this later graph.

Episode 1

To answer question 1., the students sketched the graph and wrote:

‘ The function nears 265, so, 0.1 is the rate of growth and 265 is the point of equilibrium.

The right-hand side is positive if his start C is below 265 and negative if it is above. The equilibrium point is a sink, that is, a stable equilibrium.

The students made at least one guess before they reached this result: their first try in the written report was a RK command, which was impossible to graph because the capacity of the computer-memory was exceeded. So, their strategy implied a trial-and error use of an instrumented technique that can be described as follows: 1) substitute

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[1] stands for the 4.order Runge Kutta method of numerical solution

the left side from the differential equation into the RK command, 2) type in the names of the independent and the dependent variables, 3) type in the initial values and 4) try to find values for the x-increase and the number of tangent-segments, which allows for: 5) graph the solution. Apparently, the students identified the horizontal asymptote by inspection of the graph and then graphed the function y=265 to verify the result visually. Afterwards, the equilibrium point was identified with this horizontal asymptote. So, since the graph with its asymptote was used to determine the equilibrium point, the graph with asymptote was in this case seen in a tool perspective and it was obtained using the instrumented technique sketched above.

The second part of the answer must be obtained from analysis of the differential equation. Therefore, the graphic method used in the first part of the solution serves to link graphic and analytic representations closely.

Episode 2

To answer question 2, the students wrote:

> General solution:
>
> The equation for cholesterol is of the type dy/dx=b-ay and may be solved as follows: (b is a constant)

First, the students used paper and pencil and they looked in their compendium of formulas to find the general solution. They tried to identify the type of equation.

The paper and pencil technique implied to 1) identify the type of equation, 2) recognise it in the compendium, 3) identify and substitute the actual values of the constants in the expression for the solution. The students typed the results into the computer stepwise, as they were asked to show the calculations. Apparently, they then wanted to graph the result, which is, obviously, impossible. The students used the command VECTOR to graph a family of solution curves, which could be seen as kind of an intermediate between general and specific solutions. The report reveals, that they did not completely manage this instrumented technique at that stage of their work so they must have made more than one trial: The command VECTOR(C = 265-….) would not result in graphs as shown, as far as 'C = 265…' is evaluated logically. To succeed in graphing that family of curves it is necessary to delete the 'C='.

Intermezzo

The students answered question 3 by 1) substituting the initial values in the formula for the general solution, 2) calculating the constant d and 3) substituting it into the expression. Though, the dialogue in the group revealed no clear signs of having developed a general perspective of solution to the differential equation (Andresen 2004).

Episode 3

When starting to answer the last question in this task, question 4, it was clear from the dialogue in the group that the students did not try to estimate the result, based on the preceding answers. Apparently, the fact that the students found equilibrium for the general solution earlier in the lesson did not 'ring a bell' when they were asked to argue for their latest result.The students spent some time in the group discussing how long time they had to take into account. Two of the three refused to consider the fact, that they found an asymptote.

In the final report, the students wrote:

> 'Based on the graph we conclude, that the equilibrium point does not change even if the starting point is different. The general as well as this solution therefore near to the same equilibrium point and whatever long he keeps the high level, the equilibrium point does not change.'

In the final version of the report, the students simply graphed the specific solution from episode 1 once more. Since the window was changed it is obvious that they re-graphed it. The written comment reveals that the students did not expect the coincidence between the equilibria points for the general solution and the specific one in question. This fact questions the students' adoption in advance of the general perspective on solution to differential equation. In line with this the last statement, in my interpretation, reveals unfamiliarity with the conceptions of asymptotic behaviour and of equilibrium.

Conclusion

In the case, the students' work with the task concentrated on two mathematical conceptions, represented by the example of one differential equation: 1) equilibrium point for differential equations and 2) solutions to differential equations. The equilibrium point was closely connected to asymptotic behaviour of the solution curve. So, an instrumented technique of solving and graphing the solution curve, encompassing the RK command and seeing the curve with its asymptote in a tool perspective, was used by the students to build and strengthen their conception of equilibrium point.

Determination of the general solution was carried out with a combined paper & pencil- and computer-instrumented technique, where the last part concerned change to graphic representation. Especially, the computer-instrumented part of the determination served to link between a family of solution curves, on the one hand, and the specific solution curve, examined earlier, on the other hand. The family of solution curves served as pseudo-graphing the general solution.

The experiences of asymptotic behaviour and of coincidence between the asymptotes of these solution curves, provoked by the task, supported the students' change of perspective on the two conceptions in question: Realising that 'whatever long he keeps the high level, the equilibrium point does not change' is one step to adapt an object perspective on equilibrium interpreted by horizontal asymptote. Likewise, the family of graphs are visually convincing about the fact, that the general solution should encompass the specific solution.

The case illustrates genesis of Derive-commands as an instrument in an ongoing process. The first use of RK had the character of trial and error in episode 1 (omitted from the data presented in this paper). The fact, that changes to graphic representation were not carried out with full routine, is revealed in the report in the case of VECTOR. But it was very clear, that especially the possibilities of graphing shaped the students' thinking. So, the tool influenced: 1) Their strategy, which implied to choose asymptote as the tool for finding the equilibrium, 2) Their thinking of general solution by making it tangible by pseudo-graphing into a family of solution curves and 3) Their idea of verifying the asymptotic behaviour by visual inspection and comparison with the graph of y=265.

The idea of provoking changes between tool and object perspective can be realised, for example by the asking of questions and tasks which involves ready-made procedures as well as self-developed instrumented techniques for solving modelling problems. The analysis of the case shows how the idea can facilitate proceeding of students' work as part of their learning process.

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